Inversive geometry

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The group of Euclidean transformations is denoted $E(2)$. You normally think of it as being made up of translations, rotations, and reflections. However, it can shown that every element of $E(2)$ can be represented as a combination of *reflections* alone. In inversive geometry we first study a new transformation, an *inversion,* and then use it to generate a whole group of transformations, just as reflections generate $E(2)$.

Definition[1] Given a circle centre C and radius r in $\mathbb{R}^2$, the *inversion* in this circle is the mapping $t : \mathbb{R}^2 \setminus \{C\} \to \mathbb{R}^2 \setminus \{C\}$ defined by: $t(A) = A'$, where A' lies on the straight line through C and A, and on the same side of C as A, and $CA.CA' = r^2$.

Inversions in circles behave much like reflections in straight lines: they are *not* isometries, of course, but (i) t^2 is the identity whether t stands for an inversion or a reflection, and (ii) close to the circle in which you are doing the inversion it looks very much like a reflection.

Now suppose that the circle in which we are doing the inversion is the *unit* circle (centre the origin and radius 1). Then if A has coordinates (x, y), A' must have coordinates (kx, ky), for some k, because OAA' is a straight line. The other condition is $(x^2 + y^2)(k^2x^2 + k^2y^2) = 1$. Solving for k, we find that our inversion t can be written

$$t : (x, y) \to (\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}).$$

Let us find $t(l)$, where l is the line $ax + by + c = 0$. First, let $(u, v) = t(x, y)$, so that (after a little algebra, or thought)

$$(x, y) = (\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2}).$$

Now the equation of the line is mapped to

$$\frac{au}{u^2 + v^2} + \frac{bv}{u^2 + v^2} + c = 0.$$

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[1]Unit 2 page 6 or Handbook page 62.

Relabel the coordinates by replacing (u, v) by (x, y), and multiply by $x^2 + y^2$, to obtain

$$c(x^2 + y^2) + ax + by = 0.$$

This is $t(l)$. It is a circle through the origin, as long as $c \neq 0$.

Theorem[2] Under inversion in the unit circle,

- lines though the origin map to themselves,
- lines not through the origin map to circles through the origin,
- circles through the origin map to lines not through the origin, and
- circles not through the origin map to circles not through the origin.

Now we use $\mathbb{C}$ instead of $\mathbb{R}^2$, which makes the calculations much easier. Then we define $\hat{\mathbb{C}}$, which is $\mathbb{C}$ with a point at infinity added[3], so that we no longer have to treat the origin as a special point; this is made into a geometrical construction, as opposed to merely a set-theoretic one, via *stereographic projection* to the Riemann sphere.

The *inversive group* is defined to be the group generated by all inversions in circles and all reflections in straight lines. This definition is hard to work with, though. Fortunately, we find that *Möbius transformations*[4] give us a very nice way of representing elements of the inversive group.

Definition We define a Möbius transformation $M : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$M(z) = \frac{az + b}{cz + d},$$

and if $c = 0$ then $M(\infty) = \infty$ while if $c \neq 0$ then $M(-d/c) = \infty$ and $M(\infty) = a/c$.

Theorem[5] Any element of the inversive group can be expressed as $M(z)$ or $M(\overline{z})$, where M is a Möbius transformation.

The Möbius group has many intriguing and very important ramifications, both inside and outside mathematics. For example, the Lorentz transformations in Einstein's special theory of relativity are Möbius transformations on the night sky. For more on this, and on many other fascinating things, see *The Emperor's New Mind* by Roger Penrose, Oxford University Press 1989.

[2] Unit 2 section 1.2 or Handbook page 62.
[3] Unit 2 sections 2.3 and 2.4 or Handbook pages 63 and 64.
[4] Unit 2 page 34 or Handbook page 64.
[5] Unit 2 page 40 or Handbook page 65.